Packing Anchored Rectangles

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Mentor: Tanya Khovanova

Poolesville High School

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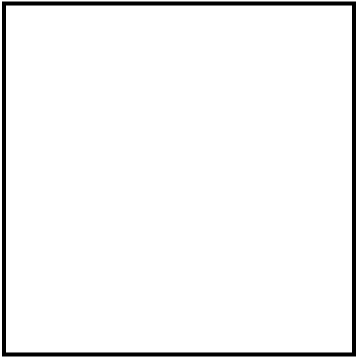
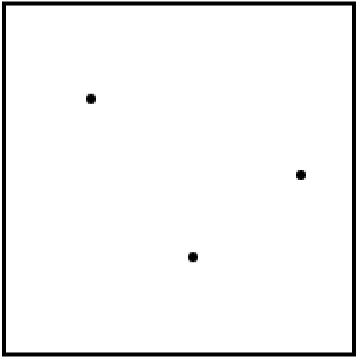
Anchored Rectangles

- Consider $[0, 1]^2$



Anchored Rectangles

- Consider $[0, 1]^2$
- n points in square



Anchored Rectangles

- Consider $[0, 1]^2$
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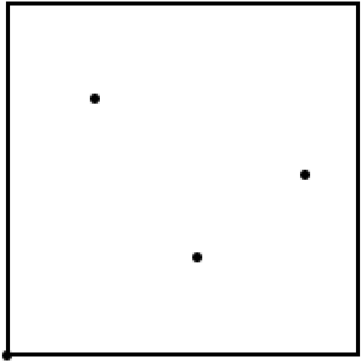
- Includes $(0, 0)$



Anchored Rectangles

- Consider $[0,1]^2$
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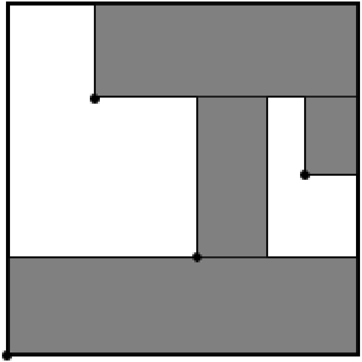
- Includes $(0,0)$

- Draw rectangles



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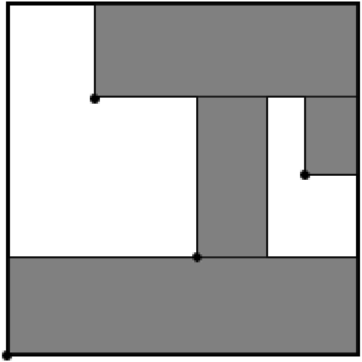

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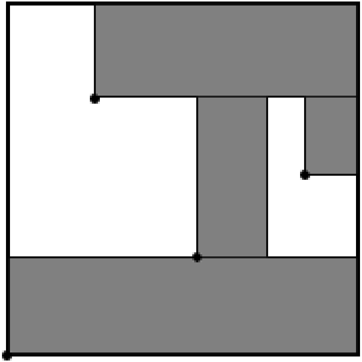
  - Rectangle has point on lower left



Anchored Rectangles

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- Draw rectangles
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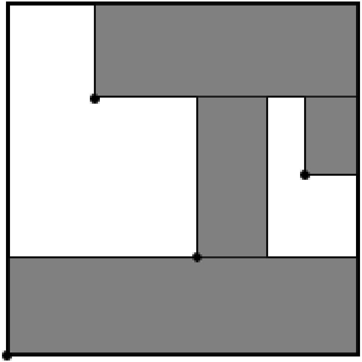
  - No overlap



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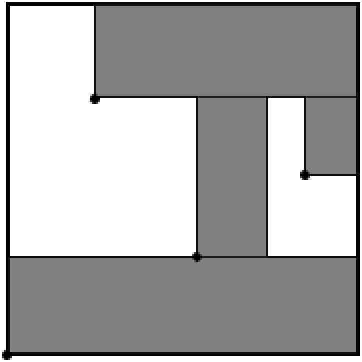
  - No points on interior



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Goal is to maximize total area of rectangles

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We can always select a set of anchored rectangles with total area at least $\frac{1}{2}$.

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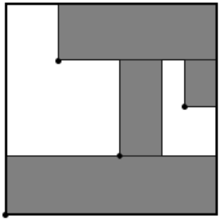

Area $= 0.6$

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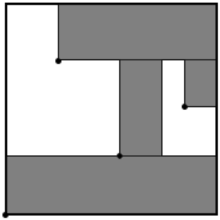

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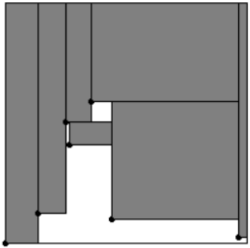

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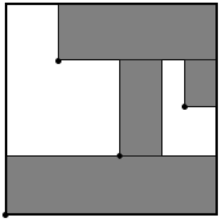

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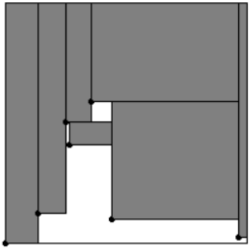

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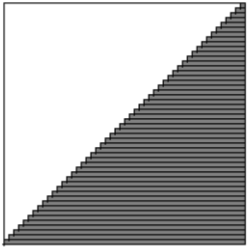

Area $= 0.51$

Past results

Equally spaced points on diagonal get arbitrarily close to $\frac{1}{2}$.

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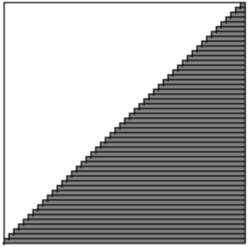

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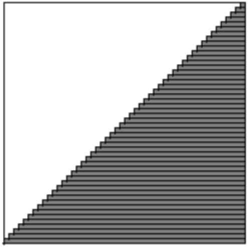

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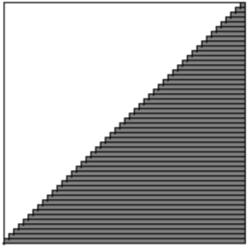

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Dumitrescu and Tóth showed greedy algorithm gets 9% in 2012.

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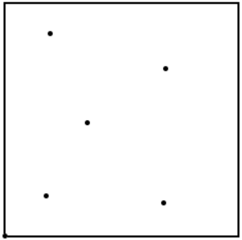

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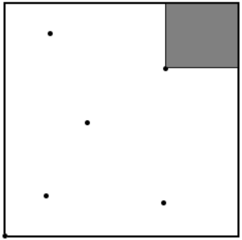

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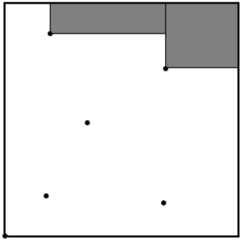

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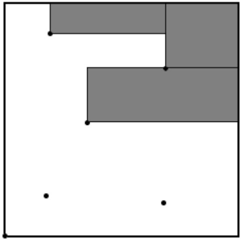

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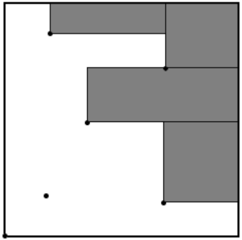

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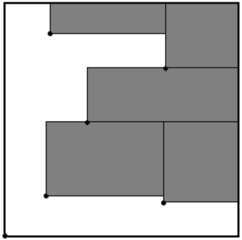

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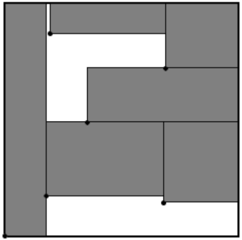

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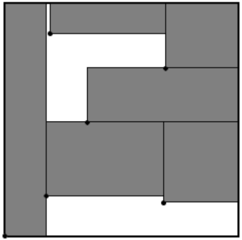

Area $= 0.7$

Maximal Anchored Rectangles

Definition

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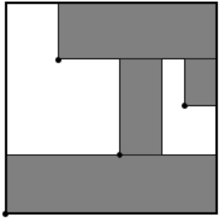

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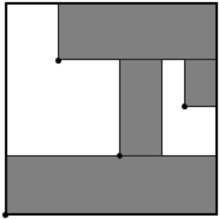
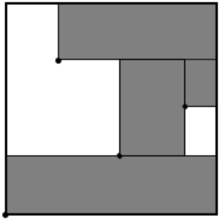


can be turned into



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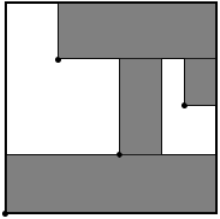

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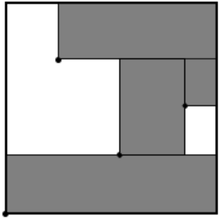
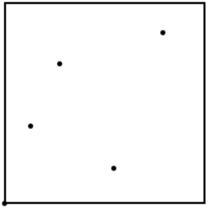


Suffices to only consider maximal rectangles

Permutations

Definition

The *permutation* of a set of points is the order of their y-coordinates when the x-coordinates are sorted



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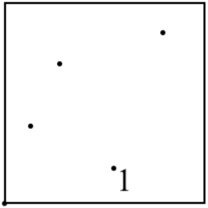

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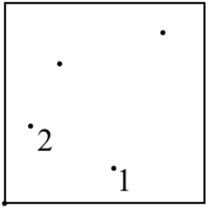

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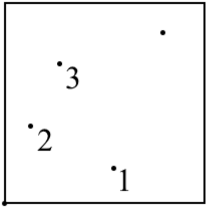

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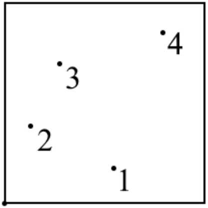

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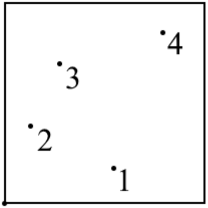

(2, 3, 1, 4)

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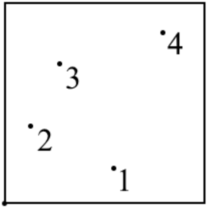


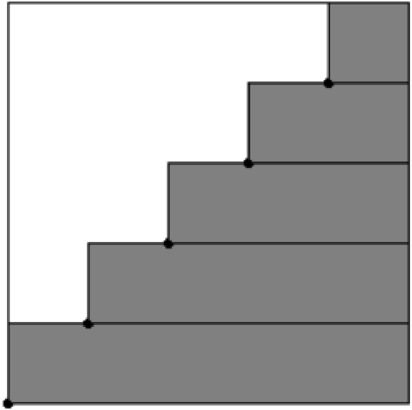
(2, 3, 1, 4)

Some permutations far easier to deal with

Types of permutations

Conjecture proved for some permutations:

- Increasing



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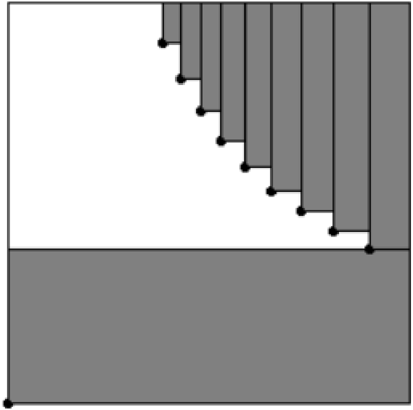

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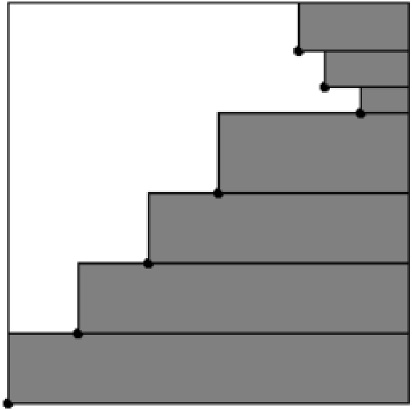
Conjecture proved for some permutations:

- Increasing
- Decreasing
- Cliff



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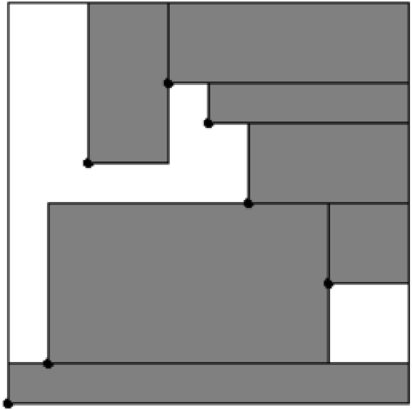
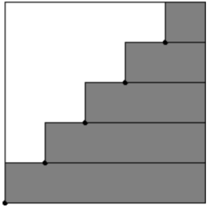
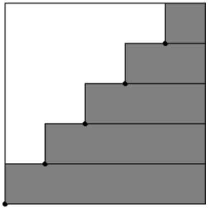


Increasing



$n = 5$

Increasing



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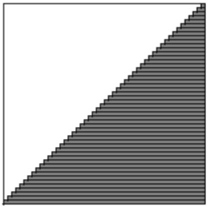
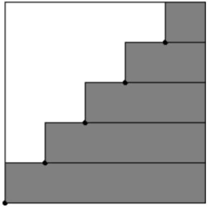


$n = 50$

Increasing



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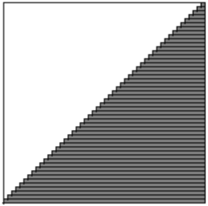

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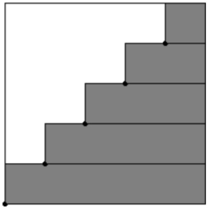
Theorem

In the increasing case with n points, we can fill at least $\frac{1}{2} + \frac{1}{2n}$ *of the square.*



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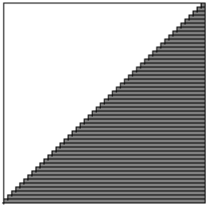

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Theorem

In the increasing case with n points, we can fill at least $\frac{1}{2} + \frac{1}{2n}$ *of the square.*
Equality iff $P_i = \left(\frac{i}{n}, \frac{i}{n}\right)$*, where* $1 \leq i \leq n-1$*.*

Scaling idea

Results hold for all rectangles

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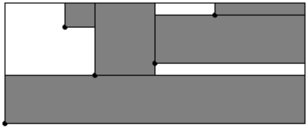

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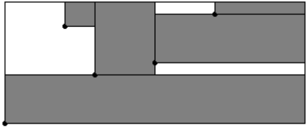

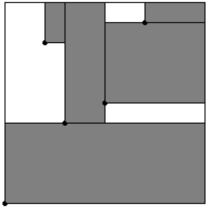

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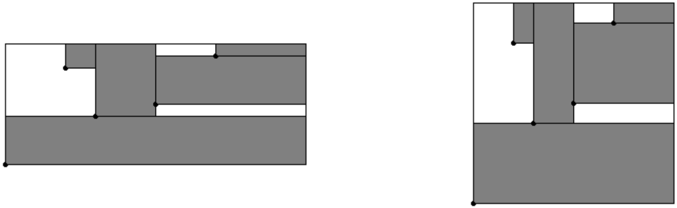
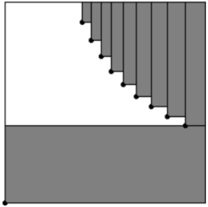


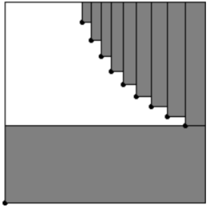
Packing density preserved

Decreasing



$n = 10$

Decreasing



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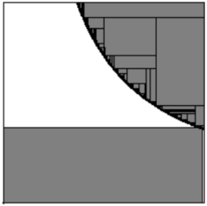
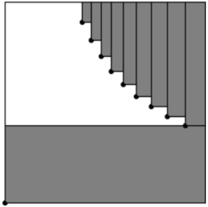


$n = 150$

Decreasing



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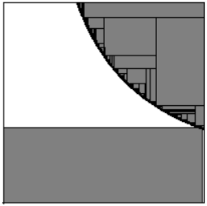
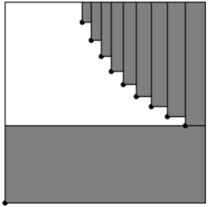


$n = 150$

Theorem

In the decreasing case with n points, we can fill at least $1 - \left(1 - \frac{1}{n}\right)^n$ *of the square.*

Decreasing



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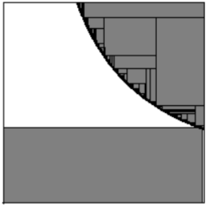
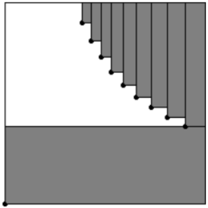


$n = 150$

Theorem

In the decreasing case with n points, we can fill at least $1 - \left(1 - \frac{1}{n}\right)^n$ *of the square.*
Equality iff $P_i = \left(\left(1 - \frac{1}{n}\right)^{n-i}, \left(1 - \frac{1}{n}\right)^{i}\right)$, *where* $1 \le i \le n - 1$.

Decreasing



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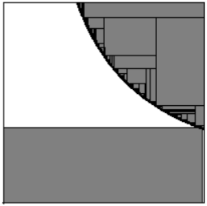

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Area approaches $1 - \frac{1}{e}$

Staircase regions

Definition

The *final decreasing run* is the maximal consecutive decreasing run that includes the rightmost points.

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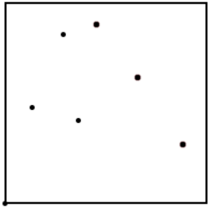

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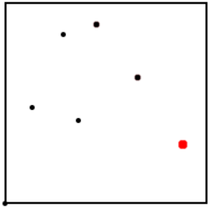

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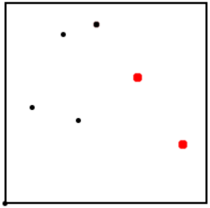

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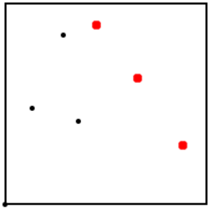

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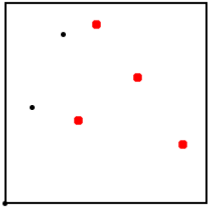

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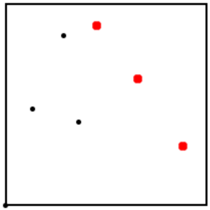

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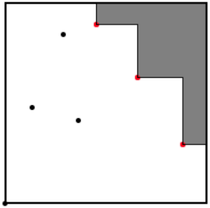

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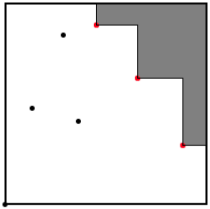

Can always fill the staircase region

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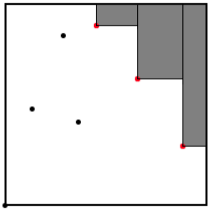

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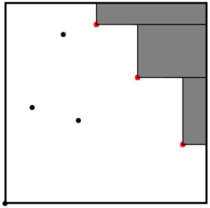

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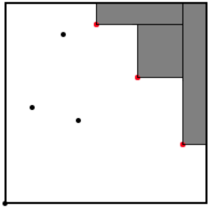

Can always fill the staircase region

Three dots

When $n = 3$, points are increasing or decreasing

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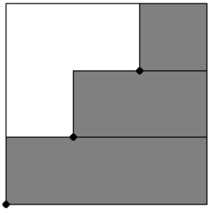

Area $\geq \frac{2}{3}$

Three dots

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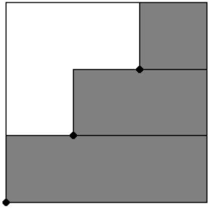
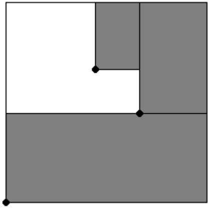


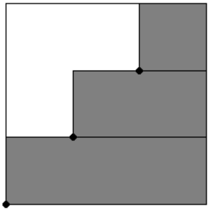
Area $\geq \frac{2}{3}$



Area $\geq \frac{19}{27}$

Three dots

When $n = 3$, points are increasing or decreasing



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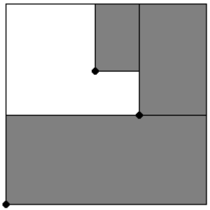
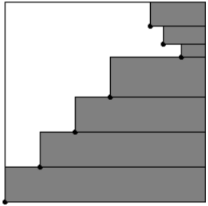


Area $\geq \frac{19}{27}$

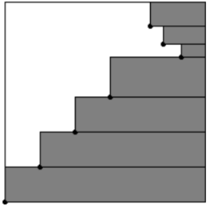
Minimum area when $n = 3$ is $\frac{2}{3} > \frac{1}{2}$.

Cliff



Area $= 0.5874$

Cliff



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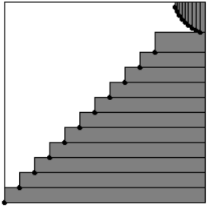
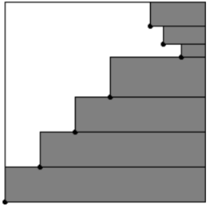


Area $= 0.5376$

Cliff



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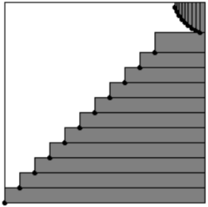
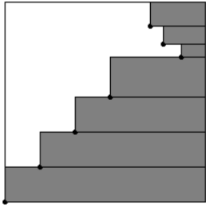


Area $= 0.5376$

Theorem

In the cliff case, we can fill more than $\frac{1}{2}$ of the square.

Cliff



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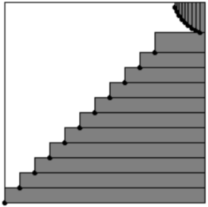

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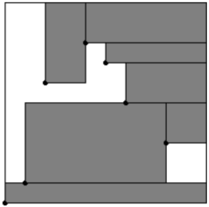
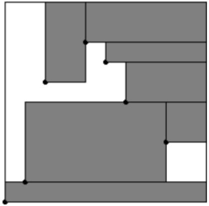
Minimum area and quality case complicated

Mountain



Area = 0.75

Mountain



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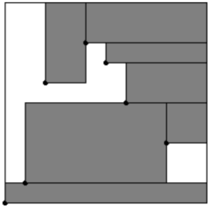

Area $= 0.75$

Theorem

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Sharper bounds and equality case currently unknown

Future Research

In the near future:

- Sharper bounds for mountain case

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- Sharper bounds for mountain case
- Consider split-layer permutations

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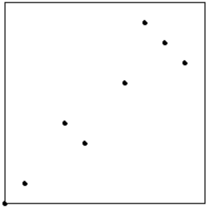

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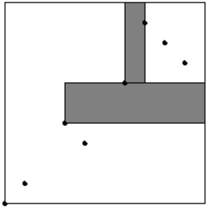

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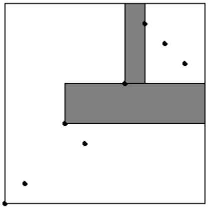

- Consider more types of permutations

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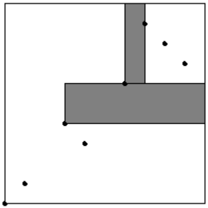

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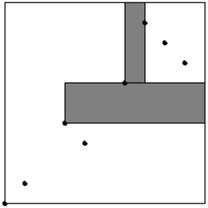

- Consider more types of permutations

In the far future:

- Prove full conjecture

Future Research

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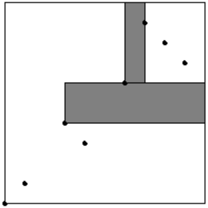


- Consider more types of permutations

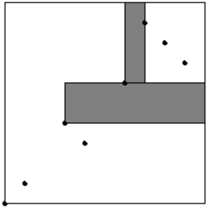
In the far future:

- Prove full conjecture
- Squares instead of rectangles

Future Research

In the near future:

- Sharper bounds for mountain case
- Consider split-layer permutations



- Consider more types of permutations

In the far future:

- Prove full conjecture
- Squares instead of rectangles
- Extend to more dimensions

Acknowledgements

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Thank you for your attention today.